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A note on Abel's partial summation formula

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Abstract. Several applications of Abel's partial summation formula to the convergence of series of positive vectors are presented. For example, when the norm of the ambient ordered Banach space is associated with a strong order unit, it is shown that the convergence of the series $\sum x_n$ implies the convergence in density of the sequence $(nx_n)_n$ to 0. This is done by extending the Koopman–von Neumann characterization of convergence in density. Also included is a new proof of the Jensen–Steffensen inequality based on Abel's partial summation formula and a trace analogue of the Tomić–Weyl inequality of submajorization.

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1. Introduction

Abel's partial summation formula (also known as Abel's transformation) asserts that every pair of families $(a_k)_{k=1}^n$, $(b_k)_{k=1}^n$ of complex numbers verifies the identity

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} \left[(a_k - a_{k+1}) \left(\sum_{j=1}^{k} b_j \right) \right] + a_n \left(\sum_{j=1}^{n} b_j \right). \qquad (Ab^{\uparrow})$$

This identity, which appears in the proof of Theorem III in [1], is instrumental in deriving a number of important results such as the Abel-Dirichlet criterion of convergence for signed series, the Abel theorem on power series, the Abel summation method (see [4,23]), Kronecker's lemma about the relationship between the convergence of infinite sums and the convergence of sequences (see [20], Lemma IV.3.2, p. 390), algorithms for establishing identities involving harmonic numbers and derangement numbers [3], the variational characterization of the level sets corresponding to majorization in $\mathbb{R}^N$ [24], Mertens' proof of his theorem on the sum of the reciprocals of primes [25] etc.

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Abel used his formula ($Ab^{\uparrow}$) through an immediate consequence of it (known as Abel's inequality): if $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $b_1, b_2, ..., b_n \in \mathbb{C}$, then

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq a_1 \max_{1 \leq m \leq n} \left|\sum_{k=1}^{m} b_k\right|.$$

Many other striking applications of this inequality may be found in the books of Pečarić, Proschan and Tong [14] and Steele [21].

Remark 1.1. Abel's formula ($Ab^{\uparrow}$) has the following backwards companion:

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} \left[(a_{k+1} - a_k)\left(\sum_{j=k+1}^{n} b_j\right)\right] + a_1 \left(\sum_{j=1}^{n} b_j\right). \tag{$Ab^{\downarrow}$}$$

A way to bring together the formulas ($Ab^{\uparrow}$) and ($Ab^{\downarrow}$) is as follows:

$$\sum_{k=1}^{n} a_k b_k = \sum_{j=1}^{k-1} \left[(a_j - a_{j+1})\left(\sum_{i=1}^{j} b_i\right)\right] + a_k \left(\sum_{i=1}^{k} b_i\right) \tag{Ab}$$

$$+ a_{k+1}\left(\sum_{i=k+1}^{n} b_i\right) + \sum_{j=k+2}^{n} \left[(a_j - a_{j-1})\left(\sum_{i=j}^{n} b_i\right)\right],$$

for any index k.

It is worth remarking that the formulas ($Ab^{\uparrow}$) and ($Ab^{\downarrow}$) (as well as Ab) extend verbatim to the context of (not necessarily commutative) bilinear maps

$$\Phi : E \times F \to G,$$

where E, F and G are three vector spaces over the same base field $\mathbb{K}$. For example, the following identities hold true for all families $(x_k)_{k=1}^n$ and $(y_k)_{k=1}^n$ of elements belonging, respectively, to E and F:

$$\sum_{k=1}^{n} \Phi(x_k, y_k) = \sum_{k=1}^{n-1} \Phi\left(x_k - x_{k+1}, \sum_{j=1}^{k} y_j\right) + \Phi\left(x_n, \sum_{j=1}^{n} y_j\right) \tag{$\Phi A1$}$$

$$= \sum_{k=1}^{n-1} \Phi\left(\sum_{j=1}^{k} x_j, y_k - y_{k+1}\right) + \Phi\left(\sum_{j=1}^{n} x_j, y_n\right) \tag{$\Phi A2$}$$

and

$$\sum_{k=1}^{n} \Phi(x_k, y_k) = \sum_{k=2}^{n} \Phi\left(x_k - x_{k-1}, \sum_{j=k}^{n} y_j\right) + \Phi\left(x_1, \sum_{j=1}^{n} y_j\right) \tag{$\Phi A3$}$$

$$= \sum_{k=2}^{n} \Phi\left(\sum_{j=k}^{n} x_j, y_k - y_{k-1}\right) + \Phi\left(\sum_{j=1}^{n} x_j, y_1\right). \tag{$\Phi A4$}$$

Moreover, these identities also work (with obvious changes) when the summation range is from m to n whenever $1 \leq m \leq n$; this represents the special case where $x_1 = \cdots = x_{m-1} = 0$ and $y_1 = \cdots = y_{m-1} = 0$.

The aim of this paper is to illustrate the formulas ($\Phi A1$)–($\Phi A4$) in the context of ordered Banach spaces. For the convenience of the reader some very basic facts concerning these spaces are recalled in the next section. Then in Sect. 3 we present applications to the convergence of series in ordered Banach spaces. Section 4 is devoted to a new short proof of the Jensen–Steffensen inequality based on Abel's partial summation formula and to an extension of this inequality to the framework of Banach lattices. Finally, in Sect. 5 we prove a trace analogue of the Tomić–Weyl inequality of submajorization.

2. Preliminaries on ordered Banach spaces

An ordered vector space is any real vector space E endowed with a convex cone E_+ (the cone of positive elements) such that

$$E_+ \cap (-E_+) = \{0\} \text{ and } E = E_+ - E_+.$$

If E is at the same time a Banach space, we call E an ordered Banach space when the following compatibility condition between the two structures is fulfilled:

$$0 \leq x \leq y \text{ implies } \|x\| \leq \|y\|.$$

The usual real Banach spaces like $\mathbb{R}^N$ (the Euclidean N-dimensional space), $C(K)$ (=the space of all continuous functions defined on a compact Hausdorff space K), the Lebesgue spaces $L^p(\mathbb{R}^N)$ (for $1 \leq p \leq \infty$), as well as their infinite dimensional discrete analogues c and ℓ^p) are endowed with order relations that behave much better. Indeed, they are all Banach lattices, that is, vector lattices (meaning there exist $\max\{x, y\}$ and $\min\{x, y\}$ for every pair of elements) plus the compatibility condition

$$|x| \leq |y| \text{ implies } \|x\| \leq \|y\|;$$

here the modulus of an element z is defined as $|z| = \max\{-z, z\}$.

The order relation in a function space is usually the point-wise one defined by

$$f \leq g \text{ if and only if } f(t) \leq g(t) \text{ for all } t;$$

this remark includes the case of $\mathbb{R}^N$, whose ordering is defined by coordinates.

A bounded linear operator $T \in L(E, F)$ acting on ordered Banach spaces is called *positive* if it maps positive elements into positive elements. Typical examples are the integration operators.

In the realm of Hilbert spaces H one encounters a rather different concept of positivity. Precisely, the Banach space $\mathcal{A}(H)$, of all bounded self-adjoint linear operators $A : H \to H$, becomes an ordered vector space when endowed with the positive cone

$$\mathcal{A}(H)_+ = \{A \in \mathcal{A}(H) : \ \langle Ax, x\rangle \geq 0 \text{ for all } x \in H\}.$$

Though this ordering does not make $\mathcal{A}(H)$ a Banach lattice, it has many nice features exploited by the spectral theory of these operators. In particular, $\mathcal{A}(H)$ is an ordered Banach space such that

$$-A \leq B \leq A \text{ implies } \ \|B\| \leq \|A\|$$

and every order bounded increasing sequence of operators has a least upper bound. Moreover, since

$$\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|,$$

we have $\|A\| \leq M$ if and only if $-M \cdot I \leq A \leq M \cdot I$, where I is the identity of H. See Simon [17].

A nice account on the basic theory of Banach lattices and positive operators may be found in the classical book of Schaefer [19], while the general theory of ordered Banach spaces is available in the books of Lacey [7] and Schaefer [18].

In the next section we shall be interested in the following special class of bilinear maps acting on ordered Banach spaces.

Definition 2.1. Suppose that E, F and G are ordered Banach spaces. A bilinear map $\Phi : E \times F \to G$ is called positive if

$$x \geq 0 \text{ and } y \geq 0 \text{ imply } \Phi(x, y) \geq 0.$$

Notice that a positive bilinear map verifies the following property of monotonicity:

$$0 \leq x_1 \leq x_2 \text{ and } 0 \leq y_1 \leq y_2 \text{ imply } \Phi(x_1, y_1) \leq \Phi(x_2, y_2).$$

Indeed, $\Phi(x_2, y_2) - \Phi(x_1, y_1) = \Phi(x_2 - x_1, y_2) + \Phi(x_1, y_2 - y_1) \geq 0$.

Using formula ($\Phi A2$) and the property of monotonicity one can easily prove the following extension of Abel's inequality:

Proposition 2.2. *Suppose that $\Phi : E \times F \to G$ is a positive bilinear map. If $m \leq \sum_{i=1}^{k} x_i \leq M$ in E (for $k = 1, ..., n$) and $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$ in F, then from formula ($\Phi A2$) it follows that*

$$\Phi(m, y_1) \leq \sum_{k=1}^{n} \Phi(x_k, y_k) \leq \Phi(M, y_1).$$

Notice also that a positive bilinear map acting on ordered Banach spaces is always bounded, which means the existence of a positive constant C such that

$$\|\Phi(x, y)\| \leq C \|x\| \|y\| \ \text{ for all } \ (x, y) \in E \times F.$$

The proof follows easily by adapting the argument of Theorem 5.5 (ii) in [18], p. 228. The smallest constant C for which the above inequality holds for all $(x, y) \in E \times F$ is called the norm of Φ and is denoted $\|\Phi\|$.

Examples of positive bilinear maps are numerous. The simplest one is the pairing $\mathbb{R}\times E \to E$, $(\alpha, x) \to \alpha x$, associated with any ordered vector space E.

If E is a Banach lattice, then the duality bilinear map $B : E \times E' \to \mathbb{R}$, given by $B(x, x') = x'(x)$ is also positive.

When E, F and G are three Banach lattices all isomorphic with $L^1(\mu)$ spaces or with $L^\infty(\mu)$ spaces, then the composition map $\Phi : L(E, F) \times L(F, G) \to L(E, G)$, $\Phi(S, T) = T \circ S$, is a positive bilinear map. See Schaefer [19], Theorem 1.5, p. 232.

The operator of convolution $(f, g) \to \int_{\mathbb{R}} f(x-y)g(y)dy$ defines a positive bilinear map on $L^1(\mathbb{R}) \times L^1(\mathbb{R})$.

Last, but not least, the trace functional defines a positive bilinear map

$$\Phi : \mathcal{A}(\mathbb{R}^N) \times \mathcal{A}(\mathbb{R}^N) \to \mathbb{R}, \quad \Phi(A, B) = \operatorname{Trace}(AB).$$

Indeed, if A and B are positive, then $A^{1/2}BA^{1/2}$ is also a positive operator and $\operatorname{Trace}(AB) = \operatorname{Trace}\left(A^{1/2}BA^{1/2}\right)$. Notice that Φ defines a scalar product on $\mathcal{A}(\mathbb{R}^N)$ whose associated norm is the Frobenius norm,

$$|||A||| = \left(\operatorname{Trace}\left(A^2\right)\right)^{1/2}.$$

This norm is equivalent to the usual operator norm on $\mathcal{A}(\mathbb{R}^N)$,

$$\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

3. Application to the convergence of positive series

Many tests of convergence for positive series extend to the framework of ordered Banach spaces as sketched in the preceding section. For example, so is the case for Olivier's test of convergence:

Theorem 3.1. *Suppose that $\Phi : E \times F \to G$ is a positive bilinear map acting on ordered Banach spaces and $(x_n)_n$ and $(y_n)_n$ are two sequences of positive elements belonging, respectively, to E and F that fulfil the following conditions:*

(a) *$(x_n)_n$ is decreasing and $\|x_n\| \to 0$;*

(b) *The series $\sum \Phi(x_n, y_n)$ is convergent.*

Then

$$\lim_{n\to\infty} \Phi\left(x_n, \sum_{k=1}^{n} y_k\right) = 0.$$

Proof. Indeed, for $\varepsilon > 0$, arbitrarily fixed, one can find an index $N > 1$ such that $\|\sum_{k=N}^{\infty} \Phi(x_k, y_k)\| < \varepsilon/2$. Then the inequalities

$$0 \leq \Phi(x_n, \sum_{k=N}^{n} y_k) \leq \sum_{k=N}^{n} \Phi(x_k, y_k) \leq \sum_{k=N}^{\infty} \Phi(x_k, y_k),$$

yield $\|\Phi(x_n, \sum_{k=N}^{n} y_k)\| < \varepsilon/2$ for every $n \geq N$. Since $\|x_n\| \to 0$ and

$$\|\Phi(x_n, y_k)\| \leq \|\Phi\| \, \|x_n\| \sup_{1 \leq k \leq N-1} \|y_k\|$$

for every $k = 1, ..., N-1$, we infer the existence of an index $\tilde{N}$ such that for every $n \geq \tilde{N}$,

$$\sup_{1 \leq k \leq N-1} \|\Phi(x_n, y_k)\| < \varepsilon/2N.$$

Therefore

$$\begin{aligned} \left\| \Phi\left(x_n, \sum_{k=1}^{n} y_k\right) \right\| &\leq \sum_{k=1}^{N-1} \|\Phi(x_n, y_k)\| + \left\| \Phi(x_n, \sum_{k=N}^{n} y_k) \right\| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for every $n \geq \tilde{N}$ and the proof is done. □

Corollary 3.2. *If $\sum x_n$ is a convergent series of positive elements in an ordered Banach space E and the sequence $(x_n)_n$ is decreasing, then $n \|x_n\| \to 0$.*

Olivier's test of convergence represents the scalar case of Corollary 3.2. In his paper from 1827, Olivier wrongly claimed that $nx_n \to 0$ is also a sufficient condition for the convergence of a numerical positive series whose terms form a sequence decreasing to 0. One year later, Abel [2] disproved this claim by considering the case of the divergent series $\sum \frac{1}{n \log n}$. See [13] for more details about this story that played an important role in rigorizing the theory of numerical series.

Theorem 3.1 allows us to derive an analogue of Abel's partial summation for series:

Theorem 3.3. *Suppose that $\Phi : E \times F \to G$ is a positive bilinear map acting on ordered Banach spaces and $(x_n)_n$ and $(y_n)_n$ are two sequences of positive elements belonging, respectively, to E and F such that $(x_n)_n$ is decreasing and $\|x_n\| \to 0$. Then the series $\sum \Phi(x_n, y_n)$ and $\sum \Phi\left(x_n - x_{n+1}, \sum_{k=1}^{n} y_k\right)$ have the same nature and in case of convergence they have the same sum,*

$$\sum_{n=1}^{\infty} \Phi(x_n, y_n) = \sum_{n=1}^{\infty} \Phi\left(x_n - x_{n+1}, \sum_{k=1}^{n} y_k\right).$$

Proof. One implication follows easily from Theorem 3.1 and Abel's partial summation formula ($\Phi A1$),

$$\sum_{k=1}^{n} B\left(x_k, y_k\right) = \sum_{k=1}^{n-1} B\left(x_k - x_{k+1}, \sum_{j=1}^{k} y_j\right) + B\left(x_n, \sum_{j=1}^{n} y_j\right).$$

Conversely, suppose the series $\sum_{n=1}^{\infty} \Phi\left(x_n - x_{n+1}, \sum_{k=1}^{n} y_k\right)$ is convergent. Then, according to our hypotheses,

$$\begin{aligned} 0 \leq \Phi\left(x_n, \sum_{k=1}^{n} y_k\right) &\leq \sum_{k=n}^{\infty} \Phi\left(x_k - x_{k+1}, \sum_{j=1}^{n} y_j\right) \\ &\leq \sum_{k=n}^{\infty} \Phi\left(x_k - x_{k+1}, \sum_{j=1}^{k} y_j\right) \end{aligned}$$

and the squeeze theorem allows us to conclude that $\Phi\left(x_n, \sum_{k=1}^{n} y_k\right) \to 0$. The proof ends with a new appeal to formula ($\Phi A1$). $\square$

Corollary 3.4. *Suppose that $\sum x_n$ is a convergent series of positive elements in an ordered Banach space E. Then the series $\sum_n \left(\sum_{k=n}^{\infty} x_k\right)$ and $\sum n x_n$ have the same nature and in case of convergence they have the same sum,*

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} x_k\right) = \sum_{n=1}^{\infty} n x_n.$$

Coming back to Olivier's test of convergence, it is worth noting that in the absence of monotonicity, only a weaker form of Corollary 3.2 holds true.

Lemma 3.5. *If $\sum x_n$ is a convergent series of positive elements in an ordered Banach space E, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} k x_k = 0.$$

Proof. Indeed, by denoting $S_n = \sum_{k=1}^{n} x_k$ for $n = 1, 2, 3, ...$, the sequence $(S_n)_n$ is convergent, say to S. According to Cesàro's theorem,

$$\lim_{n\to\infty} \frac{S_1 + \cdots + S_{n-1}}{n} = S,$$

whence

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n} = \lim_{n\to\infty}\left(S_n - \frac{S_1 + \cdots + S_{n-1}}{n}\right) = 0.$$

$\square$

If $\sum x_n$ is a convergent series of positive elements in a Banach lattice E, then for every choice of the signs $\pm$ the series $\sum \pm x_n$ is also convergent. Therefore, for every continuous linear functional $x' \in E'$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} k \left| x'\left(x_k\right)\right| = 0,$$

that is, the sequence $(nx_n)_n$ is weakly mixing to 0. See Zsidó [26] for a theory of these sequences.

Suppose now that E is an ordered Banach space with a strong order unit $u > 0$ and the norm of E is associated with the strong order unit. This means that

$$E = \bigcup_{n=1}^{\infty} [-nu, nu]$$

and

$$\|x\| = \inf \left\{\lambda > 0 : x \in [-\lambda u, \lambda u]\right\}.$$

Examples of such spaces are $C(K)$, $L^{\infty}(\mu)$, c, ℓ^{∞}, $\mathcal{A}(H)$ etc. For them one can reformulate the conclusion of Lemma 3.5 in terms of convergence in density.

Definition 3.6. A sequence $(x_n)_n$ of elements belonging to a Banach space E *converges in density* to $x \in E$ (abbreviated, (d)-$\lim\limits_{n\to\infty} x_n = x$) if for every $\varepsilon > 0$ the set $A(\varepsilon) = \{n : \|x_n - x\| \geq \varepsilon\}$ has zero density, that is,

$$\lim_{n\to\infty} \frac{|A(\varepsilon) \cap \{1, \ldots, n\}|}{n} = 0.$$

Here $|\cdot|$ stands for cardinality.

Introduced by Koopman and von Neumann in [6], this concept has proved useful in ergodic theory and its applications. See the monograph of Furstenberg [5].

The following result provides a discrete analogue of Koopman–von Neumann's characterization of convergence in density within the framework of ordered Banach spaces.

Theorem 3.7. *Suppose that E is an ordered Banach space whose norm is associated with a strong order unit $u > 0$. Then for every sequence $(x_n)_n$ of positive elements of E,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = 0 \Rightarrow (d)\text{-}\lim_{n\to\infty} x_n = 0.$$

The converse works under additional hypotheses, for example, for bounded sequences.

Proof. Assuming $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = 0$, we associate with each $\varepsilon > 0$ the set $A_\varepsilon = \{n \in \mathbb{N} : x_n \geq \varepsilon u\}$. Since

$$0 \leq \frac{|\{1,...,n\} \cap A_\varepsilon|}{n} u \leq \frac{1}{n}\sum_{k=1}^{n} \frac{x_k}{\varepsilon}$$
$$\leq \frac{1}{\varepsilon n}\sum_{k=1}^{n} x_k \to 0 \text{ as } n \to \infty,$$

we infer that each of the sets A_ε has zero density. Therefore $(d)\text{-}\lim_{n\to\infty} x_n = 0$.

Suppose now that $(x_n)_n$ is a bounded sequence and $(d)\text{-}\lim_{n\to\infty} x_n = 0$. Since boundedness in norm is equivalent to boundedness in order, there is a number $C > 0$ such that $x_n \leq Cu$ for all n. Then for every $\varepsilon > 0$ there is a set J of zero density outside which $x_n < \varepsilon u$ and we have

$$\frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{n}\sum_{k\in\{1,...,n\}\cap J} x_k + \frac{1}{n}\sum_{k\in\{1,...,n\}\setminus J} x_k$$
$$\leq \frac{|\{1,...,n\} \cap J|}{n} \cdot Cu + \varepsilon u.$$

Since $\lim_{n\to\infty} \frac{|\{1,...,n\}\cap J|}{n} = 0$, we conclude that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = 0$. ☐

Corollary 3.8. *If $\sum x_n$ is a convergent series of positive elements in an ordered Banach space E whose norm is associated with a strong order unit, then*

$$(d)\text{-}\lim_{n\to\infty} nx_n = 0.$$

Simple numerical examples show that the conclusion of Corollary 3.8 cannot be improved.

4. Connection with Jensen–Steffensen inequality

From the bilinear form of Abel's partial summation formula (see $(\Phi A1)$ and $(\Phi A3)$ above) we infer the following result that offers instances where the sum of non necessarily positive elements is yet nonnegative.

Theorem 4.1. *Suppose that E, F and G are ordered vector spaces and $\Phi : E\times F \to G$ is a positive bilinear map. If $x_1, x_2, ..., x_n \in E$ and $y_1, y_2, ..., y_n \in F$ satisfy one of the following two conditions*

$$\text{(i) } x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \textit{ and } \sum_{k=1}^{j} y_k \geq 0 \textit{ for all } j \in \{1, 2, ..., n\},$$

$$\text{(ii) } 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \textit{ and } \sum_{k=j}^{n} y_k \geq 0 \textit{ for all } j \in \{1, 2, ..., n\},$$

then

$$\sum_{k=1}^{n} \Phi\left(x_k, y_k\right) \geq 0.$$

The alert reader will recognize here the framework of another important result in real analysis, the Steffensen extension of Jensen's inequality:

Theorem 4.2. (Steffensen [22]) *Suppose that $x_1, ..., x_n$ is a monotonic family of points in an interval $[a, b]$ and $w_1, ..., w_n$ are real weights such that*

$$\sum_{k=1}^{n} w_k = 1 \quad \text{and} \quad 0 \leq \sum_{k=1}^{m} w_k \leq \sum_{k=1}^{n} w_k \quad \text{for every } m \in \{1, ..., n\}. \qquad \text{(dSt)}$$

Then every convex function f defined on $[a, b]$ verifies the inequality

$$f\left(\sum_{k=1}^{n} w_k x_k\right) \leq \sum_{k=1}^{n} w_k f(x_k). \qquad (JSt)$$

The proof of Theorem 4.2 can be easily reduced to the case of continuous convex functions and next (via an approximation argument) to the case of piecewise linear convex functions. Taking into account the following result that describes the structure of piecewise linear convex functions, the proof of Theorem 4.2 reduces ultimately to the case of the absolute value function.

Theorem 4.3. (Hardy-Littlewood-Polya) *Let $f\colon [a, b] \to \mathbb{R}$ be a piecewise linear convex function. Then f is the sum of an affine function and a linear combination, with positive coefficients, of translates of the absolute value function. In other words, f is of the form*

$$f(x) = \alpha x + \beta + \sum_{k=1}^{n} c_k |x - x_k|$$

for suitable $\alpha, \beta, x_1, ..., x_n \in \mathbb{R}$ and suitable nonnegative coefficients $c_1, \ldots, c_n$.

Simple proofs are available in [16] and [10], pp. 34-35.

Proof. (of Theorem 4.2) We have already noted that the critical case is that of the absolute value function. This can be settled as follows. Assuming the ordering $x_1 \leq \cdots \leq x_n$ (to make a choice), we infer that

$$0 \leq x_1^+ \leq \cdots \leq x_n^+$$

and

$$x_1^- \geq \cdots \geq x_n^- \geq 0$$

where $z^+ = \max\{z, 0\}$ and $z^- = \max\{-z, 0\}$ denote, respectively, the positive part and the negative part of any element z. According to Theorem 4.1 (applied to the bilinear map $B(w,x) = wx$) we have

$$\sum_{k=1}^{n} w_k x_k^+ \geq 0 \text{ and } \sum_{k=1}^{n} w_k x_k^- \geq 0,$$

equivalently

$$\left| \sum_{k=1}^{n} w_k x_k \right| \leq \sum_{k=1}^{n} w_k \left| x_k \right|,$$

and the proof is done. □

As was noted in [10], Exercise 3, p. 184, Theorem 4.3 does not extend to higher dimensions. However, there is a nontrivial class of convex functions for which Steffensen's inequality still works. Given an order interval $[u,v]$ of a Banach lattice E, let us denote by $Cv_0([u,v],E)$ the closure (in the point-wise convergence topology) of the convex cone consisting of all functions $f:[u,v] \to E$ of the form

$$f(x) = A(x) + \sum_{k=1}^{n} c_k |x - x_k|$$

for some affine function $A : E \to E$, some elements $x_1, ..., x_n \in [u,v]$ and some positive coefficients $c_1, ..., c_n$. The functions belonging to $Cv_0([u,v],E)$ satisfy the condition of convexity

$$f\left((1-\lambda)\, x + \lambda y\right) \leq (1-\lambda)\, f(x) + \lambda f(y)$$

for all $x, y \in E$ and $\lambda \in [0,1]$ (inequality taking place in the ordering of E). An inspection of the argument of Theorem 4.2 easily shows that this result still works for functions belonging to $Cv_0([u,v],E)$:

Theorem 4.4. (The generalization of the Jensen–Steffensen Inequality) *Suppose that E is a Banach lattice, $x_1, ..., x_n$ is a monotonic family of points in an order interval $[u,v]$ of E and $w_1, ..., w_n$ is a family of real weights. Then every function f belonging to $Cv_0([u,v],E)$ verifies the inequality*

$$f\left(\sum_{k=1}^{n} w_k x_k\right) \leq \sum_{k=1}^{n} w_k f(x_k).$$

5. A connection with majorization theory

The theory of majorization provides a unified approach to the analysis of a number of models in economics, finance, risk management, genetics etc. and is masterfully exposed in the remarkable book of Marshall, Olkin and Arnold [8].

Given a vector $x \in \mathbb{R}^N$ of components $x_1, ..., x_N$, let $x^{\downarrow}$ be the vector with the same entries as x but rearranged in decreasing order,

$$x_1^{\downarrow} \geq \cdots \geq x_N^{\downarrow}.$$

The vector x is *submajorized* by another vector y (abbreviated, $x \prec_w y$) if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \text{for } k = 1, ..., N$$

and *majorized* (abbreviated, $x \prec y$) if in addition

$$\sum_{i=1}^{N} x_i^{\downarrow} = \sum_{i=1}^{N} y_i^{\downarrow}.$$

The following result outlines a connection between Abel's partial summation formula and the Tomić–Weyl inequality of majorization ([10], Theorem 1.10.4, p.57):

Theorem 5.1. *Suppose that $\Phi : E \times E \to G$ is a positive bilinear map and $x_1, x_2, ..., x_n$ is a decreasing sequence of elements of E. If $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$ are two families of elements of E such that*

$$u_1 \geq u_2 \geq \cdots \geq u_n \geq 0 \text{ and } \sum_{k=1}^{j} u_k \leq \sum_{k=1}^{j} v_k \text{ for } j \in \{1, 2, ..., n\},$$

then

$$\sum_{k=1}^{n} \Phi(x_k, u_k) \leq \sum_{k=1}^{n} \Phi(x_k, v_k)$$

and

$$\sum_{k=1}^{n} \Phi(u_k, x_k) \leq \sum_{k=1}^{n} \Phi(v_k, x_k).$$

Proof. Indeed, according to ($\Phi A1$),

$$\begin{aligned}\sum_{k=1}^{n} \Phi(x_k, u_k) &= \sum_{k=1}^{n-1} \Phi\left(x_k - x_{k+1}, \sum_{j=1}^{k} u_j\right) + \Phi\left(x_n, \sum_{j=1}^{n} u_j\right) \\ &\leq \sum_{k=1}^{n-1} \Phi\left(x_k - x_{k+1}, \sum_{j=1}^{k} v_j\right) + \Phi\left(x_n, \sum_{j=1}^{n} v_j\right) \\ &= \sum_{k=1}^{n} \Phi(x_k, v_k).\end{aligned}$$

On the other hand from ($\Phi A2$) we infer that

$$\begin{aligned}\sum_{k=1}^{n} \Phi\left(u_k, x_k\right) &= \sum_{k=1}^{n-1} \Phi\left(\sum_{j=1}^{k} u_j, x_k - x_{k+1}\right) + \Phi\left(\sum_{j=1}^{n} u_j, x_n\right) \\ &\leq \sum_{k=1}^{n-1} \Phi\left(\sum_{j=1}^{k} v_j, x_k - x_{k+1}\right) + \Phi\left(\sum_{j=1}^{n} v_j, x_n\right) \\ &= \sum_{k=1}^{n} \Phi\left(v_k, x_k\right).\end{aligned}$$

$\square$

In the particular case where $E = \mathcal{A}(\mathbb{R}^N)$, $G = \mathbb{R}$ and $\Phi(A, B) = \text{Trace}\,(AB)$, Theorem 5.1 yields the inequality

$$\sum_{k=1}^{n} \text{Trace}\, A_k^2 \leq \sum_{k=1}^{n} \text{Trace}\, A_k B_k,$$

provided that the self-adjoint operators A_k and B_k satisfy the conditions

$$A_1 \geq A_2 \geq \cdots \geq A_n \geq 0 \text{ and } \sum_{k=1}^{j} A_k \leq \sum_{k=1}^{j} B_k \text{ for } j \in \{1, 2, ..., n\}.$$

Combining this with the Cauchy-Schwarz inequality,

$$\left(\sum_{k=1}^{n} \text{Trace}\, A_k B_k\right)^2 \leq \left(\sum_{k=1}^{n} \text{Trace}\, A_k^2\right)\left(\sum_{k=1}^{n} \text{Trace}\, B_k^2\right),$$

we arrive at the following trace inequality ascribed to K. L. Chung:

$$\sum_{k=1}^{n} \text{Trace}\, A_k^2 \leq \sum_{k=1}^{n} \text{Trace}\, B_k^2.$$

The function $A \to \text{Trace}\,(f(A))$ is convex on $\mathcal{A}(\mathbb{R}^N)$ whenever $f : \mathbb{R} \to \mathbb{R}$ is a convex function. See [15], Proposition 2, p. 288. Thus, Chung's inequality is an illustration of the following trace analogue of the Tomić–Weyl inequality of submajorization:

Theorem 5.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing convex function. If $A_1, A_2, ...,$ A_n and $B_1, B_2, ..., B_n$ are two families of elements of $\mathcal{A}(\mathbb{R}^N)$ such that*

$$A_1 \geq A_2 \geq \cdots \geq A_n \geq 0 \textit{ and } \sum_{k=1}^{j} A_k \leq \sum_{k=1}^{j} B_k \textit{ for } j \in \{1, 2, ..., n\},$$

then

$$\sum_{k=1}^{n} \operatorname{Trace} f\,(A_k) \le \sum_{k=1}^{n} \operatorname{Trace} f\,(B_k).$$

Proof. We will consider here the case where f is continuously differentiable. The general case can be deduced from this one by using approximation arguments. Since the function $A \to \operatorname{Trace} f(A)$ is convex on $\mathcal{A}(\mathbb{R}^N)$, for each $\lambda \in (0,1]$ we have

$$\frac{\operatorname{Trace} f\,(A+\lambda(X-A)) - \operatorname{Trace} f(A)}{\lambda} \le \operatorname{Trace} f(X) - \operatorname{Trace} f(A),$$

whence we infer (by letting $\lambda \to 0$) that

$$\operatorname{Trace}\,[f'(A)(X-A)] \le \operatorname{Trace} f(X) - \operatorname{Trace} f(A).$$

According to the bilinear form of Abel's partial summation formula ($\Phi A1$),

$$\begin{aligned}\sum_{k=1}^{n} [\operatorname{Trace} f(B_k) - \operatorname{Trace} f(A_k)] &\ge \sum_{k=1}^{n} \operatorname{Trace}\,[f'(A_k)(B_k - A_k)] \\ &= \sum_{k=1}^{n} \operatorname{Trace}\left[f'(A_n)\sum_{k=1}^{n}(B_k - A_k)\right] \\ &\quad + \sum_{m=1}^{n-1} \operatorname{Trace}\left[(f'(A_m) - f'(A_{m+1}))\sum_{k=1}^{m}(B_k - A_k)\right],\end{aligned}$$

and the right hand side is a sum of nonnegative terms due to the fact that

$$U \le V \text{ in } \mathcal{A}(\mathbb{R}^N) \text{ implies } \operatorname{Trace} h(U) \le \operatorname{Trace} h(V)$$

for all increasing and continuous functions $h : \mathbb{R} \to \mathbb{R}$. See [15], Proposition 1, p. 288. The proof ends by noting that the derivative of any continuously differentiable function is increasing and continuous. $\square$

An inspection of the argument of Theorem 5.2 shows that this result also works for nondecreasing convex functions f defined on an arbitrary interval I provided that they are continuous and the spectra of the operators A_k and B_k are included in I. The variant of Theorem 5.2 for log convex functions (such as $\operatorname{Trace}\left(e^A\right)$) can be easily obtained using the same idea.

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